5-1-1975

A Simple Characterization of Commutative Rings Without Maximal Ideals

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A SIMPLE CHARACTERIZATION OF COMMUTATIVE RINGS WITHOUT MAXIMAL IDEALS

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In a course in abstract algebra in which the instructor presents a proof that each ideal in a ring with identity is contained in a maximal ideal, it is customary to give an example of a ring without maximal ideals. The usual example is a zero-ring whose additive group has no maximal subgroups (e.g., the additive group of (dyadic) rational numbers; actually any divisible group will do; see [1, p. 67]). This may leave the impression that all such rings are artificial or at least that they abound with divisors of 0.

Below, I give a simple characterization of commutative rings without maximal ideals and a class of examples of such rings, including some without proper divisors of 0. To back up the contention that this can be presented in such a course in abstract algebra, I outline proofs of some known theorems including a few properties of radical rings in the sense of Jacobson.

The *Hausdorff maximal principle* states that every partially ordered set contains a maximal chain (i.e., a maximal linearly ordered subset). It is equivalent to the axiom of choice [4, Chapter XI].

Since the union of a maximal chain of proper ideals in a ring with identity is a maximal ideal, and since the union of a maximal chain of linearly independent subsets of a vector space is a maximal linearly independent set, we have:

(1) *Every ideal in a ring with identity is contained in a maximal ideal.*

(2) *Every non-zero vector space has a basis.*

As usual we denote the ring of integers by Z, and for any prime $p \in Z$, we denote by Z_p the ring of integers modulo p, and by Z'_p the zero-ring whose additive group is the same as that of Z_p.

It is not difficult to prove that a commutative ring R has no nonzero proper ideals if and only if either R is a field or R is isomorphic to Z'_p for some prime p. See [5, p. 133]. Hence:

(3) *An ideal M of a commutative ring R is maximal if and only if R/M is either a field or is isomorphic to Z'_p for some prime p.*

For any commutative ring R, let $J(R)$ denote the intersection of all the ideals M

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of R, such that R/M is a field. If R has no such ideals, let $J(R) = R$. In the latter case we call R a *radical ring*. The knowledgeable reader will recognize $J(R)$ as the Jacobson radical of R. See [**2**, Chapter 1].

Of the many known properties of radical rings, we need only the following two, the first of which follows immediately.

(4) *A homomorphic image of a (commutative) radical ring is a radical ring.*

(5) *$J(R)$ is a radical ring.*

Proof. If $J(R)$ is not a radical ring, then there is a homomorphism ϕ of $J(R)$ onto a field F with identity element 1. Choose $e \in J(R)$ such that $\phi(e) = 1$, and define $\phi' : R \to F$ by letting $\phi'(a) = \phi(ae)$ for each $a \in R$. If $a, b \in R$, then

$$\phi'(a + b) = \phi((a + b)e) = \phi(ae + be) = \phi(ae) + \phi(be) = \phi'(a) + \phi'(b),$$

$$\text{and } \phi'(ab) = \phi(abe) = \phi(abe)\phi(e) = \phi(aebe) = \phi(ae)\phi(be) = \phi'(a)\phi'(b).$$

Therefore ϕ' is a homomorphism of R onto F, and hence its kernel contains $J(R)$. But $e \in J(R)$ and $\phi'(e) = 1$. This contradiction shows that $J(R)$ is a radical ring.

It follows easily from (1), (3), and (4) that no ring with identity is a radical ring and that every zero-ring is a radical ring.

THEOREM. *A commutative ring R has no maximal ideals if and only if*

(a) *R is a radical ring.*

(b) *$R^2 + pR = R$ for every prime $p \in Z$.*

Proof. Suppose first that (a) and (b) hold. Since R is a radical ring, no homomorphic image of R can be a field, so, by (3) it suffices to show that for any prime $p \in Z$, the zero-ring Z_p' is not a homomorphic image of R. Suppose, on the contrary, that there is a homomorphism ϕ of R onto Z_p' with kernel M. If

$$c = \sum_{i=1}^{n} a_i b_i \in R^2, \text{ then } \phi(c) = \sum_{i=1}^{n} \phi(a_i)\phi(b_i) = 0,$$

so $R^2 \subset M$. Moreover, $\phi(pa) = p\phi(a) = 0$, so $pR \subset M$. Hence $R^2 + pR \subset M \neq R$, so (b) fails. The contradiction shows that R has no maximal ideals.

Suppose next that R has no maximal ideals. By (3) and the definition of $J(R)$, R is a radical ring. Suppose (b) fails for some prime p, let $I = R^2 + pR$, and let ϕ be the natural homomorphism of R onto R/I. If $a, b \in R$, then $0 = \phi(ab) = \phi(a)\phi(b)$, so R/I is a zero-ring, and since $0 = \phi(pa) = p\phi(a) = 0$, R/I has characteristic p and hence is a vector space over Z_p. By (2), since $I \neq R$, R/I has a basis $\{x_\alpha\}_{\alpha \in \Gamma}$ and each $x \in R/I$ may be written uniquely as $x = \sum_{\alpha \in \Gamma} a_\alpha x_\alpha$ with $a_\alpha \in Z_p$ and $a_\alpha = 0$ for all but finitely many $\alpha \in \Gamma$. For any fixed $\alpha_0 \in \Gamma$, the mapping ψ_0 such that $x\psi_0 = a_{\alpha_0}$ is a homomorphism of R/I onto Z_p'. Then $\phi \circ \psi_0$ is a homomorphism of R onto Z_p'. By (3), the kernel of $\phi \circ \psi_0$ is a maximal ideal, contrary to assumption. Hence (a) and (b) hold.

Recall that an abelian group G is *divisible* if $nG = G$ for every $n \in Z$ and note that G is divisible if and only if $pG = G$ for every prime $p \in Z$. It follows from the theorem that a zero-ring whose additive group is divisible has no maximal ideals.

COROLLARY. *Let S be a commutative ring with identity that has a unique maximal ideal R. If $R^2 + pR = R$ for every prime $p \in Z$, then R has no maximal ideals. In particular, if the additive group of S is divisible, then R has no maximal ideals.*

I conclude with some explicit examples:

Examples. (i) For a field F, let $F[x]$ denote the ring of polynomials in an indeterminate x with coefficients in F, and let $F(x)$ denote the field of quotients of $F[x]$. Let

$$S(F) = \left\{h(x) = \frac{f(x)}{g(x)} \in F(x): f(x), g(x) \in F[x] \text{ and } g(0) \neq 0\right\}.$$

It is easy to verify that $S(F)$ is an integral domain whose unique maximal ideal is $R(F) = xS(F)$. If F has characteristic zero, then, by the corollary, $R(F)$ has no maximal ideals. If F has prime characteristic, then, since $[R(F)]^2 = x^2R(F)$, the ring $R(F)$ does have maximal ideals.

(ii) Let G denote the additive semigroup of non-negative dyadic rational numbers, and let $U(F)$ denote the semigroup algebra over G with coefficients in a field F. We may regard each element of $U(F)$ as a polynomial in $x^{(\frac{1}{2})^n}$ for some positive integer n. Let $T(F)$ denote those elements of the quotient field of $U(F)$ whose denominators fail to vanish at 0. It is not difficult to verify that $R^*(F) = \{h \in T(F): h(0) = 0\}$ is the unique maximal ideal of $T(F)$ and that $[R^*(F)]^2 = R^*(F)$. By the corollary, $R^*(F)$ has no maximal ideals (and no proper divisors of 0).

(iii) Let F_1 be a field of characteristic 0, let F_2 be a field of prime characteristic p, and let R be the direct sum of the ring $R(F_1)$ described in (i) and the ring $R^*(F_2)$ described in (ii). Since each of these latter two rings is a radical ring, so is R. For, otherwise, there would be a homomorphism ϕ of R onto a field F. Then $\phi[R(F_1)]$ and $\phi[R^*(F_2)]$ are ideals of F whose (direct) sum is F, and hence one of them is all of F, contrary to the fact that $R(F_1)$ and $R^*(F_2)$ are radical rings. Also, while $R^2 \neq R$ and $pR \neq R$, it is true that $R^2 + pR = R$, so R has no maximal ideals.

One can create more rings satisfying the hypothesis of the corollary by starting with any commutative ring S with identity and divisible additive group, taking its localization S_M at a maximal ideal M, and letting $R = MS_M$. See [**1**, Chapter 2].

I am grateful to Professor Barbara Beechler for valuable criticisms of earlier drafts of this note.

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